Image Theory for Reflected TE/TM Wave in Waveguide

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***Abstract*— The image principle is extended to the time–harmonic problem of TE/TM wave propagation and reflection in a waveguide. The fictitious image generating the reflected field is derived with the aid of Heaviside operational calculus and a transmission-line model of the waveguide. The operational calculus reveals that the image of a point-like source in front of the waveguide discontinuity is another point-like source in the mirror-image position and a line source extending from the mirror-image position to infinity. The image derived with operational calculus turns out to be independent of the waveguide's transverse geometry.**

***Index Terms*—Electromagnetic analysis, electromagnetic reflection, electromagnetic scattering, transmission-line discontinuities, transmission-line theory.**

I. Introduction

PROBLEMS concerning TE or TM wave reflection in a waveguide are usually tackled with methods that are of approximate nature—the waveguide is somehow discretized and the fields are solved numerically, Huygens sources are used on the interface, etc. On the other hand, the image theory gives exact solutions, provided that the images can be found. The lack of a suitable method for finding the images is the reason that the image theory has not been widely used in waveguide problems. However, this paper proposes an easy way to derive the images, with the use of an old, but in its directness, attractive method—the Heaviside operational calculus [1], [2]. With only a few manipulation steps, the calculus gives an operational form of the image expression that can be evaluated in some cases even in a closed form. The one case where the closed-form expression is possible is the problem of the TE/TM wave reflection in a waveguide with an abrupt change in the waveguide longitudinal parameter profile. Before applying Heaviside operational calculus to this problem we must, in one way or another, cast the waveguide geometry in one-dimensional (1-D) or z-dependent form, so that the reflection operators acting on z-dependent generalized functions can be applied. With transmission-line formalism given in the Appendix, we can reduce a waveguide problem into a 1-D problem—and that is how we start in Section II. We then derive the images in Section III and finally, in Section IV, consider an example in rectangular geometry.

II. Series Expansions for the Field and the Source

Let us first find the modal representation of the scalar field f in the waveguide with cross section S and boundary ∂S. The field f is a solution of the inhomogenous Helmholtz equation

$$(\nabla^2 + k)f(\vec{r}) = g(\vec{r}) \tag{1}$$

and, furthermore, f satisfies either Neumann- or Dirichlet-type boundary conditions on ∂S. We assume that f and source g can be written as

$$f(\vec{r}) = \sum_k^\infty \sum_l^\infty \psi_{kl}(\vec{\rho}) a_{kl}(z)$$
$$g(\vec{r}) = \sum_k^\infty \sum_l^\infty \psi_{kl}(\vec{\rho}) s_{kl}(z) \tag{2}$$

where ψ's are orthonormal in the sense

$$\int_S \psi_{kl}(\vec{\rho}) \psi_{k'l'}(\vec{\rho})\, dS = \delta_{kk'}\delta_{ll'} \tag{3}$$

and satisfy the transverse eigenvalue equation

$$(\nabla_T^2 + k_{kl}^2)\psi_{kl}(\vec{\rho}) = 0. \tag{4}$$

Substituting (2) into (1) gives

$$\sum_{kl} (\nabla^2 + k^2)\psi_{kl}(\vec{\rho}) a_{kl}(z) = \sum_{kl} \psi_{kl}(\vec{\rho}) s_{kl}(z) \tag{5}$$

where Σ_{kl} denotes the double sum of (2). Operating (5) from left with $\int_S dS \psi_{mn}$ and using (3) and (4) finally yields

$$(\partial_z^2 + \beta_{mn}^2) a_{mn}(z) = s_{mn}(z) \tag{6}$$

where $\beta_{mn}^2 = k^2 - k_{mn}^2$. The assumption of the separability of f and g enables us to reduce the problem to a transmission-line one, where a_{mn} plays a role of mode voltage or current wave and s_{mn} generally represents a combined voltage and current source for mode mn. With the waveguide reflection problem in the mind, we let (6) describe the mode voltage/current $a_{mn}^i(z)$ due to source $s_{mn}^i(z)$ located in the region $z > 0$, and we then let $a_{mn}^i(z)$ be incident on interface at $z = 0$ after which the waveguide parameters abruptly change. The change in the parameters naturally gives rise to reflected mode voltage/current wave (as well as alters the incident wave traveling to the negative z-direction). The reflected modal wave can be thought of as being produced by source s_{mn}^r in the region $z < 0$ if the interface is removed and the transmission line $z > 0$ is extended to the region $z < 0$. Now the reflected-mode voltage/current is given by

$$(\partial_z^2 + \beta_{mn}^2) a_{mn}^r(z) = s_{mn}^r(z) \tag{7}$$

with the usual solution that we may try to write in the following form:

$$a_{mn}^r = R(\beta_{mn}) a_{mn}^i(0) \exp(-j\beta_{mn} z) \tag{8}$$

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where, in fact, $a^i_{mn}(0)\exp(-j\beta_{mn}z) = a^i_{mn}(-z)$ and, thus, the substitution of (8) into (7) gives

$$\begin{aligned} s^r_{mn}(z) &= (\partial_z^2 + \beta_{mn}^2)R(\beta_{mn})a^i_{mn}(-z) \\ &= R(\beta_{mn})(\partial_z^2 + \beta_{mn}^2)a^i_{mn}(-z). \end{aligned} \tag{9}$$

However, the right-hand side (RHS) of (9) is nothing else but

$$R(\beta_{mn})(\partial_z^2 + \beta_{mn}^2)a^i_{mn}(-z) = R(\beta_{mn})s^i_{mn}(-z) \tag{10}$$

because of (6)

$$(\partial_z^2 + \beta_{mn}^2)a^i_{mn}(z) = s^i_{mn}(z).$$

III. Image Source Through Heaviside Operational Calculus

A. Identification of Propagation Coefficients with Differential Operators

The RHS of (10) gives *an operational form* of the reflection image source $s^r_{mn}(z)$ as follows:

$$s^r_{mn}(z) = R(\beta_{mn})s^i_{mn}(-z). \tag{11}$$

The term *operational* could be understood by considering the exp-dependence of the solutions of (6) or (7), which allows us to identify

$$\beta_{mn}e^{-j\beta_{mn}z} = j\partial_z e^{-j\beta_{mn}z} \tag{12}$$

and by supposing that the reflection coefficient of the mode mn can be expanded as a power series of β_{mn}. The operational form of the reflection coefficient is then a pseudo-differential operator $R(j\partial_z)$, where every β_{mn} has been replaced by $j\partial_z$. As $R(\beta_{mn})$ was not a function of the indices mn, but rather of β_{mn}, the resulting $R(j\partial_z)$ does not depend on the indices mn or, in other words, $R(\beta_{mn})$s are similar in form for all modes mn and, consequently, all the modes have the same operational expression $R(j\partial_z)$ for the reflection coefficient.

Thus, if the series expansion of $R(\beta_{mn})$ is $R(\beta_{mn}) = \Sigma_k^\infty r_k(j\partial_z)^k$, the image has the expression

$$s^r_{mn}(z) = R(j\partial_z)s^i_{mn}(-z) = \sum_k^\infty r_k(j\partial_z)^k s^i_{mn}(-z). \tag{13}$$

The image source $g^r(\vec{r})$ for the total reflected field $f^r(\vec{r})$ is then

$$\begin{aligned} g^r(\vec{r}) &= \sum_{mn}\psi_{mn}(\vec{\rho})s^r_{mn}(z) \\ &= R(j\partial_z)\sum_{mn}\psi_{mn}(\vec{\rho})s^i_{mn}(-z) \\ &= R(j\partial_z)g^i(\overline{\overline{C}}\cdot\vec{r}) \end{aligned} \tag{14}$$

where $\overline{\overline{C}}$ stands for the reflection dyadic $\overline{\overline{I}} - 2\vec{u}_z\vec{u}_z$. Equation (14) actually claims that the image is independent of the waveguide transverse geometry since the guide geometry parameters do not appear in the operational expression. This rather surprising result is mathematically due to the fact that we could pull the reflection operator out of the sum in (14), but the physical reason for this can be seen if we apply the image principle also in the transverse plane. The example given in Section IV discusses the case in more detail.

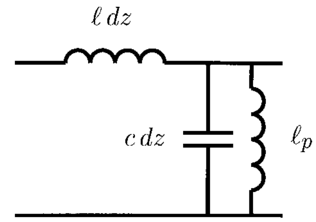

Fig. 1. Differential part of a ℓ_p transmission line modeling TE wave propagation in the waveguide.

Now the incident source $g^i(\vec{\rho})$ is usually highly localized and, thereby, the modal sources s^i_{mn} are of distributive nature. Here, the Heaviside operational calculus steps in (as the very essence of the Heaviside method is to let differential operators act on distributions [3, Ch. 1]). The fundamental identity of the Heaviside calculus is

$$\partial_z^{-\nu}\theta(z) = \frac{z^{\nu-1}}{\Gamma(\nu)}\theta(z) \tag{15}$$

whence almost all other identities can be derived. Here, $\theta(z)$ is the Heaviside unit step function. The calculus allows us to directly determine the image sources without turning to the standard integral transform methods which awkwardly lose the physical setting by forcing us to work in transform space. In the next section, the physical setting we turn our attention to is the case of the waveguide TE- and TM-mode reflection, which can be reformulated to resemble (2)–(10). The reformulation is explained in the Appendix, which recapitulates the results of the classical references [4, Ch. 1] and [5, Ch. 8]. An eigenfunction expansion of the axial magnetic or electric field leads to the concept of the mode voltage $U_{mn}(z)$ and current $I_{mn}(z)$ that satisfy the first-order transmission-line equations (70)–(73) given in the Appendix.

B. Properties of the Transmission Line

For a moment, let us suppress the indices mn from the mode voltage/current and consider only the properties of the transmission-line model for the waveguide TE/TM wave propagation. The model differs from the ordinary lossless transmission-line model by introducing either a parallel inductance element ℓ_p (TE) (Fig. 1) or a series capacitance element c_s (TM) (Fig. 2) to the differential section of the ordinary line [5]. The dimensions of these new elements are [H/m] and [F/m], respectively, and are not multiplied by a unit length dz as the normal distributed transmission-line parameters ℓ [H/m] or c [F/m]. The transmission-line equations are then in a TE case as follows:

$$\partial_z U(z) + j\omega\ell I(z) = u(z) \tag{16}$$

$$\partial_z I(z) + \left(j\omega c + \frac{1}{j\omega\ell_p}\right)U(z) = i(z). \tag{17}$$

and in a TM case:

$$\partial_z U(z) + \left(j\omega\ell + \frac{1}{j\omega c_s}\right)I(z) = u(z) \tag{18}$$

$$\partial_z I(z) + j\omega c U(z) = i(z) \tag{19}$$

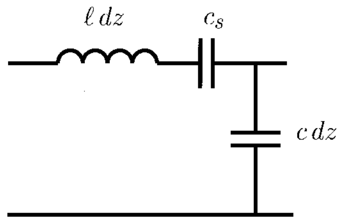

Fig. 2. Differential part of a c_s transmission line modeling TM wave propagation in the waveguide.

assuming a $e^{j\omega t}$ time variance. The source terms $u(z)$, $i(z)$ represent a distributed series voltage source and a distributed parallel current source. The propagation coefficients β are now

$$\beta = \omega\sqrt{\ell\left(c - \frac{1}{\omega^2 \ell_p}\right)} \quad \text{(TE)}$$
$$\beta = \omega\sqrt{c\left(\ell - \frac{1}{\omega^2 c_s}\right)} \quad \text{(TM)} \tag{20}$$

and corresponding characteristic admittances/impedances

$$Y = \sqrt{\frac{1}{\ell}\left(c - \frac{1}{\omega^2 \ell_p}\right)} \quad \text{(TE)}$$
$$Z = \sqrt{\frac{1}{c}\left(\ell - \frac{1}{\omega^2 c_s}\right)} \quad \text{(TM).} \tag{21}$$

It is seen that there is a relation between Z and β:

$$Z = \frac{\omega \ell}{\beta} \quad \text{(TE)} \quad Z = \frac{\beta}{\omega c} \quad \text{(TM).} \tag{22}$$

The inhomogenous Helmholtz equations for the $U(z)$ and $I(z)$ are found from (16) to (19) as follows:

$$\partial_z^2 U(z) + \beta^2 U(z) = \partial_z u(z) - j\beta Z i(z) \tag{23}$$
$$\partial_z^2 I(z) + \beta^2 I(z) = \partial_z i(z) - j\beta Y u(z). \tag{24}$$

C. $U(z)$ and $I(z)$ Generated by Sources

The solution of (23) or (24) can be written as a convolution of the source with Green function, and after partial integration the convolution integrals become

$$U(z) = \frac{1}{2}\int_{-\infty}^{\infty} [\operatorname{sgn}(z - z')u(z') + Zi(z')]e^{-j\beta|z-z'|}\, dz' \tag{25}$$

$$I(z) = \frac{1}{2}\int_{-\infty}^{\infty} [\operatorname{sgn}(z - z')i(z') + Yu(z')]e^{-j\beta|z-z'|}\, dz'. \tag{26}$$

We rewrite (25) and (26) to reveal that $U(z)$ or $I(z)$ actually describe two wavefronts traveling to opposite directions:

$$U(z) = \frac{1}{2}e^{-j\beta z}\int_{-\infty}^{z} e^{j\beta z'}[u(z') + Zi(z')]\, dz' + \frac{1}{2}e^{j\beta z}\int_{z}^{\infty} e^{-j\beta z'}[-u(z') + Zi(z')]\, dz' \tag{27}$$

$$I(z) = \frac{1}{2}e^{-j\beta z}\int_{-\infty}^{z} e^{j\beta z'}[i(z') + Yu(z')]\, dz' + \frac{1}{2}e^{j\beta z}\int_{z}^{\infty} e^{-j\beta z'}[-i(z') + Yu(z')]\, dz'. \tag{28}$$

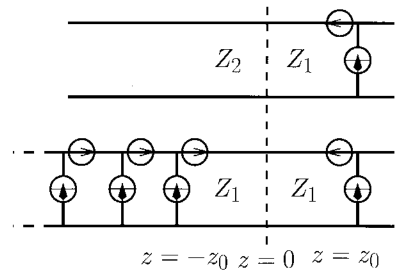

Fig. 3. Image principle in ℓ_p line. The upper section depicts the situation of the original problem with two different ℓ_p transmission lines joined at $z = 0$ and the source at $z = z_0$. In the lower section, the boundary is removed and the left-hand transmission line of the upper section is replaced by a line having the same parameters as the right-hand transmission line. However, the fields in the region $z > 0$ do not change because we have placed images in the left half-space. The image source starts from $z = -z_0$ and extends to $-\infty$.

D. Two Transmission Lines

Having established the parameters of the transmission line corresponding to the TE/TM mode propagation, we may then apply operational calculus to the problem described in the Section II—the case of mode voltage or current wave reflection and transmission from the junction of two transmission lines. The full problem splits into two parts: the TE part corresponding to the case where two ℓ_p lines with different characteristics are connected at $z = 0$ and the TM part, where two c_s lines are similarly joined together. In both cases, the line with impedance Z_1 is located in the region $z > 0$ and the line with Z_2 in $z < 0$. The source is a sum of current $i^i(z) = I_0\delta(z - z_0)$ and voltage $u^i(z) = U_0\delta(z - z_0)$ $(z_0 > 0)$ generators with infinite and zero internal impedances, respectively. The upper part of the Fig. 3 describes the situation. A current–voltage source just described is chosen because it is of the most elementary form, producing only left-going waves traveling toward the interface, provided that we set $I_0 = -U_0/Z_1$. Equations (27) and (28) then give

$$U_-^i(z) = \frac{1}{2}[-U_0 + ZI_0]e^{j\beta_1(z-z_0)}$$
$$I_-^i(z) = \frac{1}{2}[-I_0 + YU_0]e^{j\beta_1(z-z_0)}. \tag{29}$$

1) TE Modes: The reflection and transmission coefficients in the ℓ_p line corresponding to the TE case read

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{\ell_2\beta_1 - \ell_1\beta_2}{\ell_2\beta_1 + \ell_1\beta_2} \tag{30}$$
$$T = \frac{2Z_2}{Z_2 + Z_1} = \frac{2\ell_2\beta_1}{\ell_2\beta_1 + \ell_1\beta_2} \tag{31}$$

where we have used (22). In an ordinary transmission-line problem, R and T would have been independent of propagation coefficients and, consequently, the analysis would lead to almost trivial image sources, e.g., the reflected voltage image would simply be a mirror image of the original one. However, the situation depicted in (30) and (31) leads to distributed image sources as will be seen shortly. Equations (30) and (31) can be written in a form involving only either one of the propagation coefficient by requiring the transverse component of the wavenumber vector to be continuous:

$$\beta_2^2 = \beta_1^2 - B^2 \tag{32}$$

where $B^2 = \omega^2 \ell_1 c_1 - \omega^2 \ell_2 c_2$. Hence, (30) can be written in the form

$$R(\beta_1) = \frac{\ell_2\beta_1 - \ell_1\sqrt{\beta_1^2 - B^2}}{\ell_2\beta_1 + \ell_1\sqrt{\beta_1^2 - B^2}}. \quad (33)$$

Before proceeding, we note that (23) and (24) for the voltage and current are similar in form and, consequently, we only need to analyze the case of the reflected voltage wave—the current wave may be inferred by suitable changes of the variables. In order for the reflected wave to appear, the incident wave is needed:

$$(\partial_z^2 + \beta_1^2)U_-^i(z) = \partial_z u^i(z) - j\beta_1 Z_1 i^i(z) = s^i(z) \quad (34)$$

and we wish to find the image source s^r of the reflected waves given by

$$(\partial_z^2 + \beta_1)U_+^r(z) = s^r(z). \quad (35)$$

The reflected voltage can be given in the operational form

$$\begin{aligned} U_+^r(z) &= R(\beta_1)U_-^i(0)e^{-j\beta_1 z} && (36)\\ &= R(\beta_1)\,\tfrac{1}{2}\,[-U_0 + Z_1 I_0]e^{-j\beta_1(z+z_0)} \\ &= R(j\partial_z)\,\tfrac{1}{2}\,[-U_0 + Z_1 I_0]e^{-j\beta_1(z+z_0)}. && (37)\end{aligned}$$

Substituting (37) in (35) yields

$$\begin{aligned} s^r(z) &= R(j\partial_z)(\partial_z^2 + \beta_1^2)U_-^i(0)e^{-j\beta_1 z} \\ &= R(j\partial_z)(\partial_z^2 + \beta_1^2)U_-^i(-z) \\ &= R(j\partial_z)s^i(-z) && (38)\end{aligned}$$

where the last expression is the operational form of the image, which can be written as

$$\begin{aligned} R(j\partial_z)s^i(-z) &= R(j\partial_z)(-\partial_z u^i(-z) - j\beta_1 Z_1 i^i(-z)) \\ &= R(j\partial_z)(-\partial_z U_0\delta(-(z+z_0)) - j\beta_1 Z_1 \\ &\quad \cdot I_0\delta(-(z+z_0))).\end{aligned}$$

In operating on the source term s^i with R we need the formula

$$\frac{\alpha\partial_z - \sqrt{\partial_z^2 + B^2}}{\alpha\partial_z + \sqrt{\partial_z^2 + B^2}}\delta(z) = \frac{\alpha - 1}{\alpha + 1}\delta(z) + B\Upsilon(\alpha, Bz)\theta(z) \quad (39)$$

where [2], [6, pp. 238–239]

$$\Upsilon(\alpha, z) = -\frac{8\alpha}{\alpha^2 - 1}\sum_{n=1}^{\infty}\left(\frac{\alpha - 1}{\alpha + 1}\right)^n n\frac{J_{2n}(z)}{z} \quad (40)$$

and $\theta(z)$ denotes the Heaviside unit step function (see Fig. 4). A word of caution concerning the substitution of the $j\partial_z$ in radicals as $\beta_2 = \sqrt{\beta_1^2 - B^2}$ is in place here: we must choose the sign of the radical so that the result makes sense physically. Unfortunately, we have no way of knowing the sign other than by examining which one makes more sense physically. We can, for instance, examine what happens when $\ell_1 \to \ell_2$. In this case, the inspection reveals that $\sqrt{(j\partial_z)^2 - B^2} = -j\sqrt{\partial_z^2 + B^2}$. The ambiguity arises from the fact that there is no unique image for the reflected field even though the

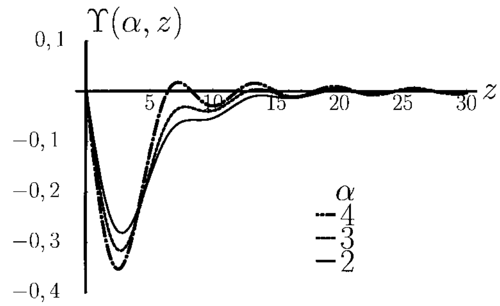

Fig. 4. Function $\Upsilon(\alpha, z)$ forming the continuous part of the RHS of (39) with different values of α.

reflected field itself must have a unique solution in the region $z > 0$.

Thus, the expression for the voltage image is

$$s^r(z) = \partial_z u^r(z) - j\beta_1 Z_1 i^r(z) \quad (41)$$

where

$$\begin{aligned} u^r(z) = -U_0\Bigg[&\frac{\ell_2 - \ell_1}{\ell_2 + \ell_1}\delta(z+z_0) + B\Upsilon\left(\frac{\ell_2}{\ell_1}, -B(z+z_0)\right) \\ &\cdot \theta(-(z+z_0))\Bigg] && (42)\\ i^r(z) = I_0\Bigg[&\frac{\ell_2 - \ell_1}{\ell_2 + \ell_1}\delta(z+z_0) + B\Upsilon\left(\frac{\ell_2}{\ell_1}, -B(z+z_0)\right) \\ &\cdot \theta(-(z+z_0))\Bigg]. && (43)\end{aligned}$$

From (41) to (43), it is seen that the reflection images are generally combined of poles and continuous line sources, where the line sources start from $z = -z_0$ and extend to $z = -\infty$ (see Fig. 3).

The complete treatment of the problem requires a transmission image too, with the image now being found through the use of the operator (31). Applying the property (32) yields for the transmission operator

$$T(\beta_2) = \frac{2\ell_2\sqrt{\beta_2^2 + B^2}}{\ell_1\beta_2 + \ell_2\sqrt{\beta_2^2 + B^2}}.$$

We assumed that the reflected voltage can be given by (35). Likewise, we assume that the transmitted voltage is a solution of

$$(\partial_z^2 + \beta_2^2)U_-^t(z) = s^t(z) = \partial_z u^t(z) - j\beta_2 Z_2 i^t(z). \quad (44)$$

To simplify the manipulations appearing in the steps to follow we introduce a "modified" incident wave:

$$(\partial_z^2 + \beta_2^2)U_-^{\prime i}(z) = \partial_z U_0\delta(z - z_0) - j\beta_2 Z_2 I_0'\delta(z - z_0)$$

where $I_0' = [Z_1(\beta_1)/Z_2(\beta_2)]I_0$, and which has a solution

$$U_-^{\prime i}(z) = \tfrac{1}{2}\,(-U_0 + Z_2 I_0')e^{j\beta_2(z-z_0)}. \quad (45)$$

Requiring $U_-^t(z)$ to be continuous across the interface gives

$$U^t(z) = T(\beta_2)U_-^i(0)e^{j\beta_2 z}$$

which combined with (44) and (45) gives

$$\begin{aligned}(\partial_z^2+\beta_2^2)T(\beta_2)(-U_0+Z_1I_0)e^{-j\beta_1 z_0}e^{j\beta_2 z}\\ =T(\beta_2)e^{-j\beta_1 z_0}e^{j\beta_2 z_0}(\partial_z^2+\beta_2^2)[-U_0+Z_2I_0'(\beta_2)]e^{j\beta_2(z-z_0)}\\ =T(\beta_2)e^{-j\beta_1 z_0}e^{j\beta_2 z_0}[\partial_z U_0\delta(z-z_0)-j\beta_2 Z_2 I_0'(\beta_2)\delta(z-z_0)]\\ =T(\beta_2)e^{-j\sqrt{\beta_2^2+B^2}\,z_0}[\partial_z U_0\delta(z)-j\beta_2 Z_2 I_0'(\beta_2)\delta(z)] \qquad (46)\end{aligned}$$

where we have used the Heaviside shifting property $e^{j\partial_z z_0}\delta(z-z_0)=\delta(z)$. We note that (44) and (46) should describe the same source s^t. The relation (46) needs only a suitable replacement for β_2's. The apparent choice for the replacement is $-j\partial_z$ because the transmitted wave is going left. Therefore,

$$\begin{aligned}u^t(z)\\ &=T(-j\partial_z)U_0e^{-j\sqrt{\beta_2^2+B^2}z_0}\delta(z)\\ &=U_o\frac{2\ell_2\sqrt{\partial_z^2-B^2}}{\ell_1\partial_z+\ell_2\sqrt{\partial_z^2-B^2}}e^{-\sqrt{\partial_z^2-B^2}z_0}\delta(z)\\ &=\left(-\frac{\partial}{\partial z_0}\right)U_o\left[\frac{2\ell_2}{\ell_1+\ell_2}J_0(jB\sqrt{z^2-z_0^2})\theta(z-z_0)\right.\\ &\quad+\frac{4\ell_1/\ell_2}{\ell_1^2/\ell_2^2-1}\sum_{n=1}^{\infty}\left(\frac{\ell_1-\ell_2}{\ell_1+\ell_2}\right)^2\left(\frac{z-z_0}{z+z_0}\right)^2\\ &\quad\left.\cdot J_{2n}(jB\sqrt{z^2-z_0^2})\theta(z-z_0)\right]\end{aligned}$$

and

$$\begin{aligned}i^t(z)\\ &=T(-j\partial_z)I_0'(-j\partial_z)e^{-j\sqrt{\beta_2^2+B^2}z_0}\delta(z)\\ &=I_o\frac{2\ell_1\partial_z}{\ell_1\partial_z+\ell_2\sqrt{\partial_z^2-B^2}}e^{-j\sqrt{\partial_z^2-B^2}z_0}\delta(z)\\ &=\left(-\frac{\partial}{\partial z_0}\right)I_o\left[\frac{2\ell_1}{\ell_1+\ell_2}J_0(jB\sqrt{z^2-z_0^2})\theta(z-z_0)\right.\\ &\quad-\frac{4\ell_1/\ell_2}{\ell_1^2/\ell_2^2-1}\sum_{n=1}^{\infty}\left(\frac{\ell_1-\ell_2}{\ell_1+\ell_2}\right)^2\left(\frac{z-z_0}{z+z_0}\right)^2\\ &\quad\left.\cdot J_{2n}(jB\sqrt{z^2-z_0^2})\theta(z-z_0)\right].\end{aligned}$$

2) TM Modes: We may instantly write down solutions corresponding to the c_s line image problem by using the solutions of the TE problem if the following dualities between TE and TM transmission lines are recognized:

$$U(z)\leftrightarrow I(z)\quad u(z)\leftrightarrow i(z)\quad \ell\leftrightarrow c\quad \ell_p\leftrightarrow c_s.$$

E. Waveguide Images

The images we have considered earlier were derived with respect to general properties of the transmission line. If they represent modal images s^r_{mn}, the entire image for the reflected H_z field is then a sum of these modal TE images:

$$\begin{aligned}g^{r,e}(\vec r)&=\sum_m^{\infty}\sum_n^{\infty}\frac{k_{mn}^2}{j\omega\ell_1}\psi^e_{mn}(\vec\rho)s^{r,e}_{mn}(z)\\ &=R(j\partial_z)\sum_m^{\infty}\sum_n^{\infty}\frac{k_{mn}^2}{j\omega\ell_1}\psi^e_{mn}(\vec\rho)s^{i,e}_{mn}(-z)\\ &=R(j\partial_z)g^{i,e}(\overline{\overline{C}}\cdot\vec r) \qquad (47)\end{aligned}$$

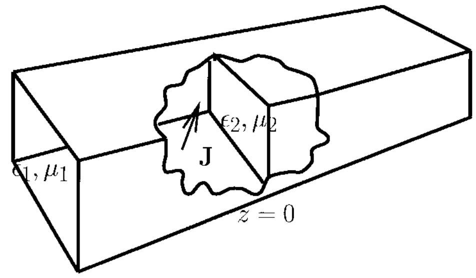

Fig. 5. Two waveguides joined at $z=0$. The waveguides are filled with materials having different permittivities and permeabilities.

and similarly, using the dualities given in (Section III-D.2), we get the image for E_z:

$$\begin{aligned}g^{r,m}(\vec r)&=-R(j\partial_z)\sum_m^{\infty}\sum_n^{\infty}\frac{k_{mn}^2}{j\omega\ell_1}\psi^m_{mn}(\vec\rho)s^{i,m}_{mn}(-z)\\ &=-R(j\partial_z)g^{i,m}(\overline{\overline{C}}\cdot\vec r). \qquad (48)\end{aligned}$$

Superscripts e and m in (47) and (48) refer to TE and TM modes, respectively. Looking at the final operational relations on the RHS of (47) and (48), we note that the eigenfunction expansions do not appear in the expressions even though we did use ψ_{mn}'s in the intermediate steps. The disappearance of the eigenfunctions can be expected since the images should be independent of the choice of the eigenfunction basis. However, even with this property, the eigenfunctions of the empty waveguide are a necessary evil, for without them we are not able to model the waveguide as a transmission line. Therefore, the image theory of this paper cannot be applied to the waveguide problem where the cross section is so complicated that the eigenfunctions are not known.

IV. JUNCTION OF RECTANGULAR WAVEGUIDES

As an example, we now consider a situation where a small dipole is placed in front of a junction of rectangular waveguides (Fig. 5). The source is

$$\vec J(\vec r)=\vec u I_s L\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\exp(j\omega t) \qquad (49)$$

and, hence, for TE modes from (64) in the Appendix we get

$$\begin{aligned}g^{i,e}(\vec r)&=\vec u_x\cdot\vec u I_s L\delta(x-x_0)\partial_y\delta(y-y_0)\delta(z-z_0)\\ &\quad-\vec u_y\cdot\vec u I_s L\partial_x\delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \qquad (50)\end{aligned}$$

Unlike the sources considered in the earlier sections, the dipole produces both left- and right-going waves of which the former gives rise to the reflection image (47)

$$\begin{aligned}g^{r,e}(\vec r)&=I_sL\vec u\cdot\Big[\vec u_x\delta(x-x_0)\partial_y\delta(y-y_0)-\vec u_y\partial_x\\ &\quad\cdot\delta(x-x_0)\delta(y-y_0)\Big]R(j\partial_z)\delta(z-z_0)\\ &=I_sL\vec u\cdot\Big[\vec u_x\delta(x-x_0)\partial_y\delta(y-y_0)-\vec u_y\partial_x\delta(x-x_0)\\ &\quad\cdot\delta(y-y_0)\Big]\times\left[\frac{\ell_2-\ell_1}{\ell_2+\ell_1}\delta(z+z_0)\right.\\ &\quad+B\Upsilon\left(\frac{\ell_2}{\ell_1},-B(z+z_0)\right)\\ &\quad\left.\cdot\theta(-(z+z_0))\right]. \qquad (51)\end{aligned}$$

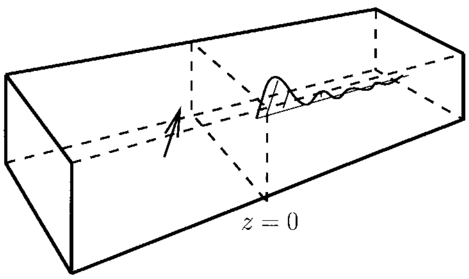

Fig. 6. The original and the image source (line image part only shown).

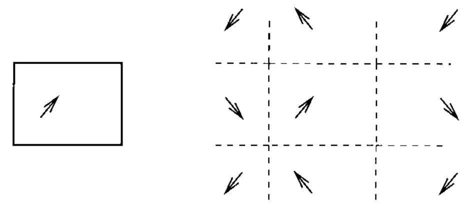

Fig. 7. (a) Rectangular waveguide's walls can be removed if infinite number of image sources similar to the original one are substituted in the (b) transverse mirror-image positions.

The source (see Fig. 6) of the z-component of the reflected $\vec{H}$-field is then a combination of: 1) a point source located at $x = x_0$, $y = y_0$, $z = -z_0$ and is otherwise similar to the original source, but the amplitude is modified by the factor $(\ell_2 - \ell_1)/(\ell_2 + \ell_1)$ and 2) a line source starting from $x = x_0$, $y = y_0$, $z = -z_0$ and extending to the infinity in the negative z-direction. The line source has an amplitude variation given by $B\Upsilon(\ell_2/\ell_1, -B(z + z_0))$.

In the case of the rectangular guide, there is a nice explanation for the independence of (51) of the transverse coordinates x, y—we can picture that in addition to the original image there is an infinite set of images placed in the mirror-image points in the transverse plane of the guide (Fig. 7). Images in the transverse plane extend longitudinally to the infinity in a similar fashion as the original one (51).

V. Conclusion

Solving waveguide discontinuity problems such as the one demonstrated here usually involve the construction of the Green function for the waveguide. The geometry of the discontinuity has to be built into the Green function in order for the Green function to satisfy boundary conditions. Work required to develop the Green function for the simple example presented in this paper might be within reasonable limits, but certainly not for more complicated settings such as scatterers in the waveguide near the discontinuity. The image-theory approach needs only an empty waveguide's Green function, and the previous problem of forming the Green function satisfying boundary conditions reduces to the case of finding the images. Once the images sources are found with the procedure described in the previous sections, the fields are immediately obtained by simply taking the convolution of all sources with the Green function of the empty waveguide. The practicability of this approach is seen clearly if, for instance, we are writing a program to calculate fields in waveguide scattering problems. We need write only one simple subroutine for the Green function no matter the number and shape of the scatterers. Now the principal work is to form the image sources, and here the Heaviside calculus may be applied. The way Heaviside calculus was used in the previous sections can be classified as a direct-type formulation of the general image theory. The other class of general image-theory formulations can be termed as being of the integral transform type. An example of the latter category is the exact image theory (EIT) that used Laplace transform techniques to find the images [7]. The disadvantage of the EIT was that the manipulations of integrals of Laplace transforms could be quite tedious. But, as we have seen in this paper, Heaviside calculus gives operational expressions for the images immediately and deals naturally with the distributive nature of the sources. A particularly attractive feature is that Heaviside calculus leads to physically sound images in the sense that the images are combinations of point-like sources that could be called "static," existing even when $\omega \to 0$, and line sources that could be thought of as being "dynamic" additions to the "static" images. After all, intuition shows that time–harmonic images should differ from the time-independent ones, yet the images should bear some resemblance to the images of the static problem.

Appendix
Transmission-Line Model for TE/TM Modes

A. Mode Voltages and Currents

In a waveguide TE- or TM-mode propagation case, the electric and magnetic fields can be conveniently expressed with scalar transverse eigenfunction ψ and scalar longitudinal eigenfunction Ω [5, Ch. 8], and for a single mode's electric and magnetic field we have the following relations, where the double mode indices mn are suppressed:

$$H_z^e(x,y,z) = \frac{k_c^2}{j\omega\mu}\psi^e(x,y)\Omega^e(z) \tag{52}$$

$$\vec{H}_t^e = \frac{1}{j\omega\mu}(\nabla_t\psi^e)\partial_z\Omega^e \tag{53}$$

$$\vec{E} = \vec{E}_t^e = (\vec{u}_z \times \nabla_t\psi^e)\Omega^e. \tag{54}$$

(55)

The TM case is dual to the previous one, hence,

$$E^m(x,y,z) = \frac{k_c^2}{j\omega\epsilon}\psi^m(x,y)\Omega^m(z) \tag{56}$$

$$\vec{E}_t^m = \frac{1}{j\omega\epsilon}(\nabla_t\psi^m)\partial_z\Omega^m \tag{57}$$

$$\vec{H} = \vec{H}^m = -(\vec{u}_z \times \nabla_t\psi^m)\Omega^m. \tag{58}$$

Now connecting $\Omega(z)$ with $U(z)$ or $I(z)$, (52)–(58) appear as

$$\vec{E}^e = \vec{e}^e U^e \qquad \vec{H}_t^e = \vec{h}^e I^e \tag{59}$$

$$\vec{E}_t^m = \vec{e}^m U^m \quad \vec{H}^m = \vec{h}^m I^m \tag{60}$$

where mode voltages or currents and mode vectors are

$$\vec{e}^e = \vec{u}_z \times \nabla_t \psi^e \quad U^e = \Omega^e \quad \vec{h}^e = -\nabla_t \psi^e$$
$$I^e = -\frac{1}{j\omega\mu}\partial_z \Omega^e \quad \vec{e}^m = -\nabla_t \psi^m \quad U^m = -\frac{1}{j\omega\epsilon}\partial_z \Omega^m$$
$$\vec{h}^m = -\vec{u}_z \times \nabla_t \psi^m \quad I^m(z) = \Omega^m. \qquad (61)$$

The mode vectors thus expressed are orthogonal and are assumed to be normalized over the cross-sectional area of guide.

B. Sources

The relation between transmission-line sources $u(z)$, $i(z)$ and current $\vec{J}(z)$ or magnetic current $\vec{M}(z)$ remains to be found. To this end we start with Helmholtz equations for $\vec{H}$, $\vec{E}$:

$$(\nabla^2 + k^2)\vec{H} = j\omega\epsilon(\vec{M} + \frac{1}{k^2}\nabla\nabla \cdot \vec{M}) - \nabla \times \vec{J} \qquad (62)$$
$$(\nabla^2 + k^2)\vec{E} = j\omega\mu(\vec{J} + \frac{1}{k^2}\nabla\nabla \cdot \vec{J}) + \nabla \times \vec{M} \qquad (63)$$

and proceed by taking the z-components of the aforementioned quantities, since it is possible to express TE or TM fields as functions of longitudinal fields only:

$$(\nabla^2 + k^2)H_z = j\omega\epsilon\left(M_z + \frac{1}{k^2}\nabla \cdot \partial_z \vec{M}\right) - \vec{u}_z \cdot \nabla \times \vec{J} = g^e(\vec{r}) \qquad (64)$$
$$(\nabla^2 + k^2)E_z = j\omega\mu\left(J_z + \frac{1}{k^2}\nabla \cdot \partial_z \vec{J}\right) + \vec{u}_z \cdot \nabla \times \vec{M} = g^m(\vec{r}). \qquad (65)$$

Substituting the series expansions

$$H_z = \sum_k^\infty \sum_l^\infty \frac{k_{kl}^2}{j\omega\mu}\psi_{kl}^e(\vec{\rho})U_{kl}^e(z)$$
$$g^e = \sum_k^\infty \sum_l^\infty \frac{k_{kl}^2}{j\omega\mu}\psi_{kl}^e(\vec{\rho})s_{kl}^e(z) \qquad (66)$$
$$E_z = \sum_k^\infty \sum_l^\infty \frac{k_{kl}^2}{j\omega\epsilon}\psi_{kl}^m(\vec{\rho})I_{kl}^m(z)$$
$$g^m = \sum_k^\infty \sum_l^\infty \frac{k_{kl}^2}{j\omega\epsilon}\psi_{kl}^m(\vec{\rho})s_{kl}^m(z) \qquad (67)$$

(where $k_{kl} = k_c$ is the cutoff wavenumber of the mode kl) in (64)–(65), multiplying from the left by ψ_{mn}, and integrating over the cross section S eventually yields

$$(\partial_z^2 + \beta_{mn}^2)U_{mn}^e(z) = s_{mn}^e(z)$$
$$(\partial_z^2 + \beta_{mn}^2)I_{mn}^m(z) = s_{mn}^m(z). \qquad (68)$$

C. Circuit Components

It is seen from (61) that one may define mode impedances

$$Z_{mn}^e = \frac{\omega\mu}{\beta_{mn}} \quad Z_{mn}^m = \frac{\beta_{mn}}{\omega\epsilon}. \qquad (69)$$

Equation (61), combined with (68), finally yield the transmission-line equations with sources

$$\partial_z U_{mn}^e + j\beta_{mn} Z_n^e I_{mn}^e = \partial_z U_n^e + j\omega\mu I_{mn}^e = 0 \qquad (70)$$
$$\partial_z I_{mn}^e + j\beta_{mn} Y_{mn}^e U_{mn}^e = \partial_z I_{mn}^e + \left(j\omega\epsilon + \frac{k_{cn}^2}{j\omega\mu}\right)U_{mn}^e = -\frac{1}{j\omega\mu}s_{mn}^e(z) = i_{mn}^e(z) \qquad (71)$$

for TE and

$$\partial_z U_{mn}^m + j\beta Z_{mn}^m I_{mn}^m = \partial_z U_{mn}^m + \left(j\omega\mu + \frac{k_c^2}{j\omega\epsilon}\right)I_{mn}^m = -\frac{1}{j\omega\epsilon}s_{mn}^m(z) = u_{mn}^m(z) \qquad (72)$$
$$\partial_z I_{mn}^m + j\beta Y_{mn}^m U_{mn}^m = \partial_z I_{mn}^m + j\omega\epsilon U_{mn}^m = 0 \qquad (73)$$

for TM. Comparing (70) and (71) with (16) and (17), and (72) and (73) with (18) and (19), one recognizes the following correspondences:

$$c \Leftrightarrow \epsilon \quad \ell \Leftrightarrow \mu \quad c_s \Leftrightarrow \epsilon/k_c^2 \quad \ell_p \Leftrightarrow \mu/k_c^2. \qquad (74)$$

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